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The Reissner-Sagoci problem for a non-homogeneous half-space with a penny-shaped crack

H.T. DANYLUK, B.M. SINGH and J. VRBIK*
*Department of Mechanical Engineering, University of Saskatchewan, Saskatoon, Saskatchewan, S7N 0W0, Canada and *Department of Mathematics, Brock University St. Catherines, Ontario, Canada, L2S 3A1*

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Abstract. The present paper examines the elastostatic problem related to the axisymmetric rotation of a rigid circular disc bonded to a non-homogeneous half-space containing a penny-shaped crack. The shear modulus of the half-space is assumed to vary with depth according to the relation $\mu(z) = \mu_1(z + c)^\alpha, c > 0$ and μ_1, α are constants. Using Hankel transforms, the solution of the problem is reduced to integral equations and finally to simultaneous Fredholm integral equations of the second kind. By solving numerically the simultaneous Fredholm integral equations, results are obtained which are used to estimate the stress intensity factor at the crack tip and the torque required to rotate the disc through an angle ω_0.

1. Introduction

The statical Reissner-Sagoci problem [1, 2, 3, 4, 5] is that of determining the components of stress and displacement in the interior of a semi-infinite homogeneous isotropic solid $z \geqslant 0$ when a circular region $(0 \leqslant r \leqslant a, z = 0)$ of the boundary surface is forced to rotate through an angle α about an axis normal to the undeformed plane surface of the solid. The problem of the torsion of an elastic cylinder embedded in an elastic half-space with a different modulus of rigidity was considered by Dhaliwal, Singh and Sneddon [6]. The statical problem of the torsion of an annular disk attached to a semi-infinite cylinder embedded in an elastic half-space was considered by Singh *et al.* [7].

For a non-homogeneous, isotropic elastic half-space, the Reissner-Sagoci problem has been solved recently by Kassir [8] assuming the modulus of rigidity of the medium to be of the form $\mu(z) = \mu_\alpha z^\alpha (0 \leqslant \alpha \leqslant 1)$ where μ_α is a real constant and z is the coordinate perpendicular to the plane boundary of the half-space $z \geqslant 0$. The same problem has been solved by Chuaprasert and Kassir [9] by assuming a more realistic form for the modulus of rigidity, namely, $\mu(z) = \mu_0(c + z)^\alpha$, where μ_0, c and α are real constants. Dhaliwal and Singh [10] considered the Reissner-Sagoci problem for an isotropic non-homogeneous elastic layer $(0 \leqslant z \leqslant h)$ of finite thickness h and modulus of rigidity $\overline{\mu}_1(z) = \mu_1(a + z)^\alpha$ perfectly bonded to an isotropic non-homogeneous elastic half-space $(z \geqslant h)$ of modulus of rigidity $\overline{\mu}_2(z) = \mu_2(b + z)^\beta$ where $\mu_1, \mu_2, a, b, \alpha$ and β are real constants. Selvadurai, Singh and Vrbik [11] examine the Reissner-Sagoci problem for a half-space with shear modulus of the form $G(z) = G_1 + G_2 e^{-\xi z}$, where G_1, G_2 and ξ are real constants. References to static and dynamic torsion problems are found in the book by Gladwell [12]. In this paper we consider the torsion of an elastic non-homogeneous half-space by a rigid disk. The non-homogeneous half-space contains a penny-shaped crack at the depth $z = h$ from the plane surface of the disk. The shear modulus of the half-space is assumed to vary with the depth according to the relation $\mu(z) = \mu_1(c + z)^\alpha, c > 0$ and μ_1, α are constants. With the aid of Hankel transforms, an exact formulation for the mixed boundary value problem is presented in the form of integral equations. The solution of these integral equations is reduced to two simultaneous Fredholm integral equations of the second kind. After solving numerically two simultaneous Fredholm integral equations, the numerical results for the torque required to rotate the disk and the stress intensity factor at the tip of the crack are obtained and results are displayed graphically. The corresponding problem of the torsion of a homogeneous elastic half-space with a crack by a disk has been solved by Low [16] and by Kassir and Sih [17].

The motivation of this problem lies in the area of crack mechanics and the paper also contains a novel treatment for a crack problem involving a non-homogeneous half-space having practical value in soil mechanics.

2. Basic equation and its solution

In the case of an axisymmetric torsion problem, the displacement vector U assumes the form $(0, v, 0)$ in the cylindrical polar coordinate system (r, θ, z) with $v = v(r, z)$. The only non-zero components of stress are given by

$$\sigma_{\theta z}(r, z) = \mu \frac{\partial v}{\partial z}, \quad \sigma_{r\theta} = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \tag{1}$$

where $\mu = \mu(z)$ is the modulus of rigidity of the material. The equation of equilibrium for a non-homogeneous, isotropic, elastic medium is given by

$$\frac{\partial}{\partial r} \sigma_{r\theta} + \frac{\partial}{\partial z} \sigma_{\theta z} + \frac{2}{r} \sigma_{r\theta} = 0 \tag{2}$$

Substituting equation (1) into equation (2), we find that the equation of equilibrium takes the form:

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{\mu} \frac{\partial \mu}{\partial z} \frac{\partial v}{\partial z} = 0 \tag{3}$$

We assume that the shear modulus μ is of the form

$$\mu(z) = \mu_1 (c + z)^{\alpha}, c > 0 \tag{4}$$

where μ_1, c and α are all real constants and c is positive. By the method of separation of variables, a suitable solution of equation (3) for $0 \leqslant z \leqslant h$ can be written as

$$v(r, z) = (c + z)^p \int_0^{\infty} \{A(\xi) K_p[(c + z)\xi] + B(\xi) I_p[(c + z)\xi]\} J_1(r\xi) \, d\xi \tag{5}$$

where

$$p = \frac{1 - \alpha}{2}, \tag{6}$$

J_q is the Bessel function of the first kind of order q and I_p and K_p are the modified Bessel functions of the first and second kind, respectively, of order p and $A(\xi)$ and $B(\xi)$ are arbitrary functions of ξ. With the help of equations (1) and (4), we find from equation (5) that

$$\sigma_{\theta z}(r, z) = -\mu_1 (c+z)^{1-p} \int_0^{\infty} \xi \{A(\xi) K_{p-1}[(c+z)\xi] - B(\xi) I_{p-1}[(c+z)\xi]\} J_1(r\xi) \, d\xi, \tag{7}$$

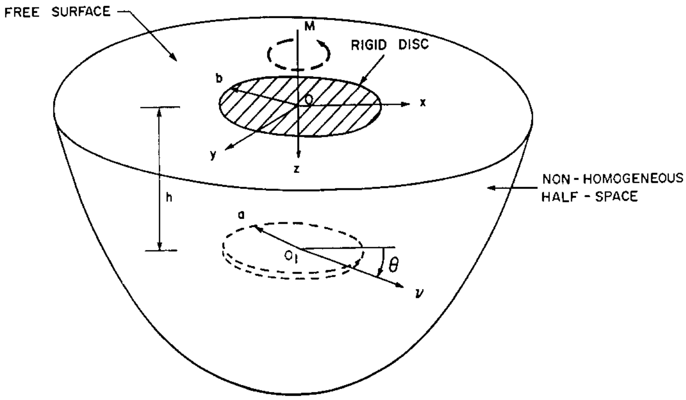

Fig. 1. Penny-shaped crack parallel to free surface due to rotation of rigid disk.

$$\sigma_{r\theta}(r,z) = -\mu_1(c+z)^{1-p}\int_0^\infty \xi\{A(\xi)K_p[(c+z)\xi] + B(\xi)I_p[(c+z)\xi]\}J_2(r\xi)\,\mathrm{d}\xi. \quad (8)$$

In the same way we find for the region $h \leqslant z < \infty$ that

$$v(r,z) = (c+z)^p\int_0^\infty C(\xi)K_p[(c+z)\xi]J_1(r\xi)\,\mathrm{d}\xi, \quad (9)$$

$$\sigma_{\theta z}(r,z) = -\mu_1(c+z)^{1-p}\int_0^\infty \xi C(\xi)K_{p-1}[(c+z)\xi]J_1(r\xi)\,\mathrm{d}\xi, \quad (10)$$

$$\sigma_{r\theta}(r,z) = -\mu_1(c+z)^{1-p}\int_0^\infty \xi C(\xi)K_p[(c+z)\xi]J_2(r\xi)\,\mathrm{d}\xi \quad (11)$$

3. Statement of the problem and derivation of the integral equations

We consider that a rigid disk of radius b is bonded to the boundary $z = 0$ of the non-homogeneous semi-infinite solid and is centered through the z-axis as shown in Fig. 1. The disk is rotated by an angle ω_0 such that

$$v(r,0) = \omega_0 r, \quad 0 \leqslant r \leqslant b \quad (12)$$

$$\sigma_{\theta z}(r,o) = 0, \quad r > b \quad (13)$$

The problem is to find the stress distribution in the non-homogeneous semi-infinite solid which contains a penny-shaped crack of radius a whose plane is at a distance h from the base of the rigid disk. In order to maintain the disk in the twisted position, a couple of magnitude M must be applied to the disk. Since the crack surfaces are free from tractions, the stress components $\sigma_{\theta z}$ must vanish as

$$\sigma_{\theta z}(r,h^+) = \sigma_{\theta z}(r,h^-) = 0, \quad 0 \leqslant r \leqslant a. \quad (14)$$

On the plane $z = h$ and for $r > a$, continuity of displacement and stress requires that

$$v(r,h^+) = v(r,h^-), \quad r > a \quad (15)$$

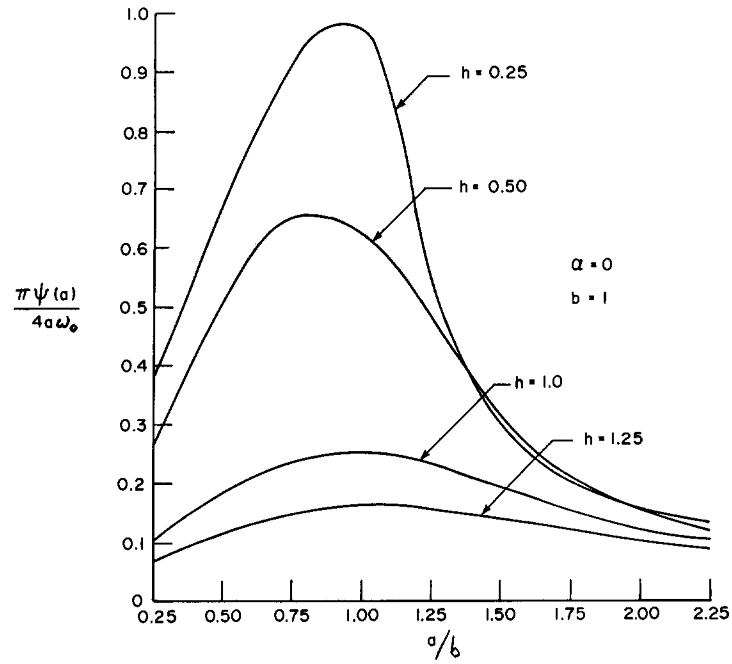

Fig. 2. Variation of $\frac{\pi\psi(a)}{4a\omega_0}$ with $\frac{a}{b}$ for $h = 0.25, 0.50, 1.0, 1.25$ with $\alpha = 0, b = 1$.

$$\sigma_{\theta z}(r, h^+) = \sigma_{\theta z}(r, h^-), \quad r > a. \tag{16}$$

With the help of equations (7) and (10), we find from the boundary conditions (14) and (16) that

$$C(\xi) = \frac{A(\xi)K_{p-1}[(c+h)\xi] - B(\xi)I_{p-1}[(c+h)\xi]}{K_{p-1}[(c+h)\xi]} \tag{17}$$

With the help of equations (5), (7), (9), (10), and (17), we find using boundary conditions (12), (13), (14) and (15), the following integral equations:

$$\int_0^\infty [A(\xi)K_p(c\xi) + B(\xi)I_p(c\xi)]J_1(r\xi)\,\mathrm{d}\xi = \frac{\omega_0 r}{c^p}, 0 \leqslant r \leqslant b \tag{18}$$

$$\int_0^\infty \xi[A(\xi)K_{p-1}(c\xi) - B(\xi)I_{p-1}(c\xi)]J_1(r\xi)\,\mathrm{d}\xi = 0, \quad r > b \tag{19}$$

$$\int_0^\infty \xi[A(\xi)K_{p-1}[(c+h)\xi] - B(\xi)I_{p-1}[(c+h)\xi]]J_1(r\xi)\,\mathrm{d}\xi = 0, \quad 0 \leqslant r \leqslant a \tag{20}$$

$$\int_0^\infty \frac{B(\xi)J_1(r\xi)\,\mathrm{d}\xi}{(c+h)\xi K_{p-1}[(c+h)\xi]} = 0, \quad r > a. \tag{21}$$

In getting equation (21) we used the result that

$$I_{p-1}[(c+h)\xi]K_p[(c+h)\xi] + I_p[(c+h)\xi]K_{p-1}[(c+h)\xi] = \frac{1}{(c+h)\xi}. \tag{22}$$

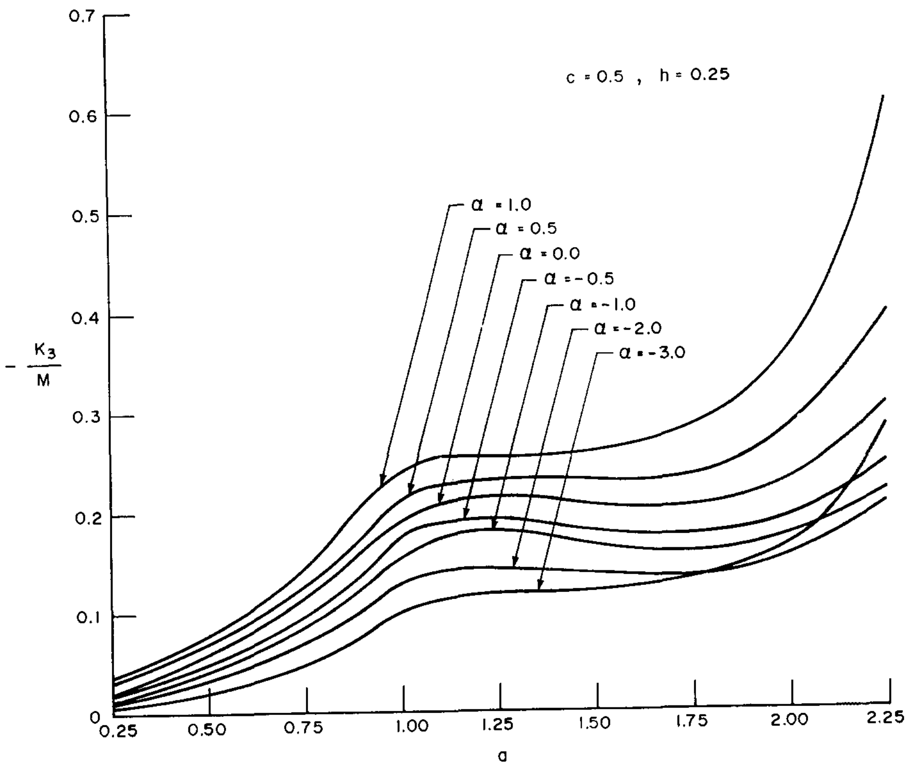

Fig. 3. Variation of $\frac{-K_3}{M}$ with a for $\alpha = 1.0, 0.5, 0.0, -0.5, -1.0, -2.0, -3.0$ and $c = 0.5, h = 0.25$.

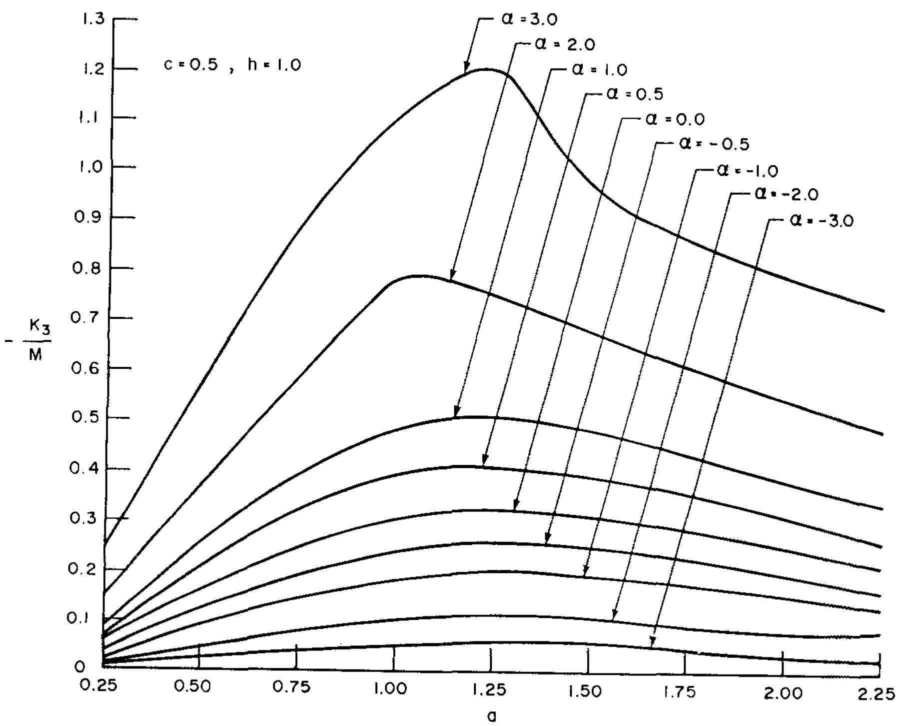

Fig. 4. Variation of $\frac{-K_3}{M}$ with a for $\alpha = 3.0, 2.0, 1.0, 0.5, 0.0, -0.5, -1.0, -2.0, -3.0$ and $c = 0.5, h = 1.0$.

4. Solution of the integral equations and expressions for physical quantities

For solving the integral equations (18), (19), (20), and (21), we will use the method of Copson [13] and assume that

$$A(\xi)K_{p-1}(c\xi) - B(\xi)I_{p-1}(c\xi) = \frac{1}{\sqrt{c}}\int_0^b \phi(t)\sin(\xi t)\,\mathrm{d}t \tag{23}$$

$$B(\xi) = (c+h)^{1/2}\xi K_{p-1}[(c+h)\xi]\int_0^a \psi(t)\left[\frac{\sin(\xi t)}{\xi t} - \cos(\xi t)\right]\mathrm{d}t \tag{24}$$

where $\phi(t)$ and $\psi(t)$ are unknown functions to be determined from the integral equations (18), (19), (20), and (21). Using integration by parts, equation (23) can be written as

$$A(\xi)K_{p-1}(c\xi) - B(\xi)I_{p-1}(c\xi) = \frac{1}{\sqrt{c}}\frac{1}{\xi}\left[-\phi(b)\cos(b\xi) + \int_0^b \cos(\xi t)\phi'(t)\,\mathrm{d}t\right] \tag{25}$$

where the prime denotes differentiation with respect to t and

$$\lim_{t\to 0}\phi(t) = 0.$$

Equation (24) can be further reduced using the result that

$$\sqrt{\frac{\pi\xi t}{2}}J_{3/2}(\xi t) = \frac{\sin(\xi t)}{\xi t} - \cos(\xi t) \tag{26a}$$

to the following form:

$$B(\xi) = \sqrt{\frac{\pi(c+h)}{2}}\,\xi^{3/2}K_{p-1}[(c+h)\xi]\int_0^a \sqrt{t}\psi(t)J_{3/2}(\xi t)\,\mathrm{d}t. \tag{26b}$$

On integrating by parts equation (24), we find that

$$B(\xi) = \sqrt{(c+h)}K_{p-1}[(c+h)\xi]\left\{-\psi(a)\sin(a\xi) + \int_0^a \frac{\sin(\xi t)}{t}\,[t\psi(t)]'\,\mathrm{d}t\right\} \tag{26c}$$

where again the prime denotes differentiation with respect to t and

$$\lim_{t\to 0}[t\psi(t)] = 0.$$

Substitution of equations (25) and (26c) into the equations (19) and (21) separately show that equations (19) and (21) are satisfied identically. To show this use is made of Bessel function integrals found in Watson [14, section 13.42] or Gradshteyn and Ryzhik [18, section 6.693]. With the help of equation (23), equation (20) can be written in the form:

$$-\int_0^\infty \frac{B(\xi)J_1(r\xi)\,\mathrm{d}\xi}{2(c+h)K_{p-1}[(c+h)\xi]} + \int_0^\infty \xi B(\xi)\left\{\frac{K_{p-1}[(c+h)\xi]I_{p-1}(c\xi)}{K_{p-1}(c\xi)} - I_{p-1}[(c+h)\xi] + \frac{1}{2[(c+h)\xi]K_{p-1}[(c+h)\xi]}\right\}J_1(r\xi)\,\mathrm{d}\xi$$
$$+\frac{1}{\sqrt{c}}\int_0^b \phi(u)\int_0^\infty \frac{\xi K_{p-1}[(c+h)\xi]}{K_{p-1}(c\xi)}\sin(\xi u)J_1(r\xi)\,\mathrm{d}\xi\,\mathrm{d}u = 0, \quad 0 \leqslant r \leqslant a. \tag{27}$$

Substituting the value of $B(\xi)$ from equation (26c) into the first integral of (27), and using the integral relationships found in Watson [14, section 13.42], one gets that

$$-\frac{1}{2r\sqrt{c+h}}\int_0^r \frac{[t\psi(t)]'\,\mathrm{d}t}{\sqrt{r^2-t^2}} + \int_0^\infty \xi B(\xi)\left\{\frac{K_{p-1}[(c+h)\xi]I_{p-1}(c\xi)}{K_{p-1}(c\xi)} - I_{p-1}[(c+h)\xi] + \frac{1}{2[(c+h)\xi]K_{p-1}[(c+h)\xi]}\right\}J_1(r\xi)\,\mathrm{d}\xi$$
$$+\frac{1}{\sqrt{c}}\int_0^b \phi(u)\int_0^\infty \frac{\xi K_{p-1}[(c+h)\xi]}{K_{p-1}(c\xi)}\sin(\xi u)J_1(r\xi)\,\mathrm{d}\xi\,\mathrm{d}u = 0, \quad 0 \leqslant r \leqslant a. \tag{28}$$

This equation is of Abel type. Hence its solution can be written using results from Sneddon [15, pp. 41-42], together with the aid of equations (26a), (26b) and the result from Watson [14, section 12.11] that

$$\int_0^t \frac{r^2 J_1(r\xi)\,\mathrm{d}r}{(t^2-r^2)^{1/2}} = \left(\frac{\pi}{2\xi}\right)^{1/2} t^{3/2} J_{3/2}(\xi t) \tag{29}$$

to give

$$\psi(t) - 2\int_0^b L_1(u,t)\phi(u)\,\mathrm{d}u + \int_0^a K_1(u,t)\psi(u)\,\mathrm{d}u = 0, \quad 0 < t < a, \tag{30}$$

where

$$L_1(u,t) = \frac{2}{\pi}\sqrt{\frac{c+h}{c}}\int_0^\infty \frac{K_{p-1}[(c+h)\xi]}{K_{p-1}(c\xi)}\sin(\xi u)\left[\frac{\sin(\xi t)}{\xi t} - \cos(\xi t)\right]\mathrm{d}\xi, \tag{31}$$

$$\begin{aligned} K_1(u,t) &= -2\sqrt{ut}\int_0^\infty (c+h)\xi^2 K_{p-1}[(c+h)\xi] \\ &\times\left\{\frac{I_{p-1}(c\xi)K_{p-1}[(c+h)\xi]}{K_{p-1}(c\xi)} - I_{p-1}[(c+h)\xi] + \frac{1}{2\xi(c+h)K_{p-1}[(c+h)\xi]}\right\} \\ &\times J_{3/2}(u\xi)J_{3/2}(t\xi)\,\mathrm{d}\xi \end{aligned} \tag{32}$$

Substituting the value of $A(\xi)$ from equation (23) into equation (18), we find that

$$\frac{1}{r\sqrt{c}}\int_0^r \frac{t\phi(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}} + \frac{1}{\sqrt{c}}\int_0^b \phi(t)\int_0^\infty \left(\frac{K_p(c\xi)}{K_{p-1}(c\xi)} - 1\right)\sin(\xi t)J_1(r\xi)\,\mathrm{d}\xi\,\mathrm{d}t$$
$$+\int_0^\infty \frac{B(\xi)J_1(r\xi)\,\mathrm{d}\xi}{(c\xi)K_{p-1}(c\xi)} = \frac{\omega_0 r}{c^p}, o < r < b \tag{33}$$

In getting equation (33), we used the result given by equation (22) and the relationship from Watson [14, section 13.42] that

$$\int_0^\infty J_1(r\xi)\sin(\xi t)\,\mathrm{d}\xi = \frac{tH(r-t)}{r\sqrt{r^2-t^2}} \tag{34}$$

where $H(r-t)$ denotes the Heaviside function. Equation (33) is also of Abel type. Hence its solution can be written again by using results from Sneddon [15, pp. 41-42] together with equation (26b) and the results that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{r^3\,\mathrm{d}r}{(t^2-r^2)^{1/2}} = 2t^2 \tag{35}$$

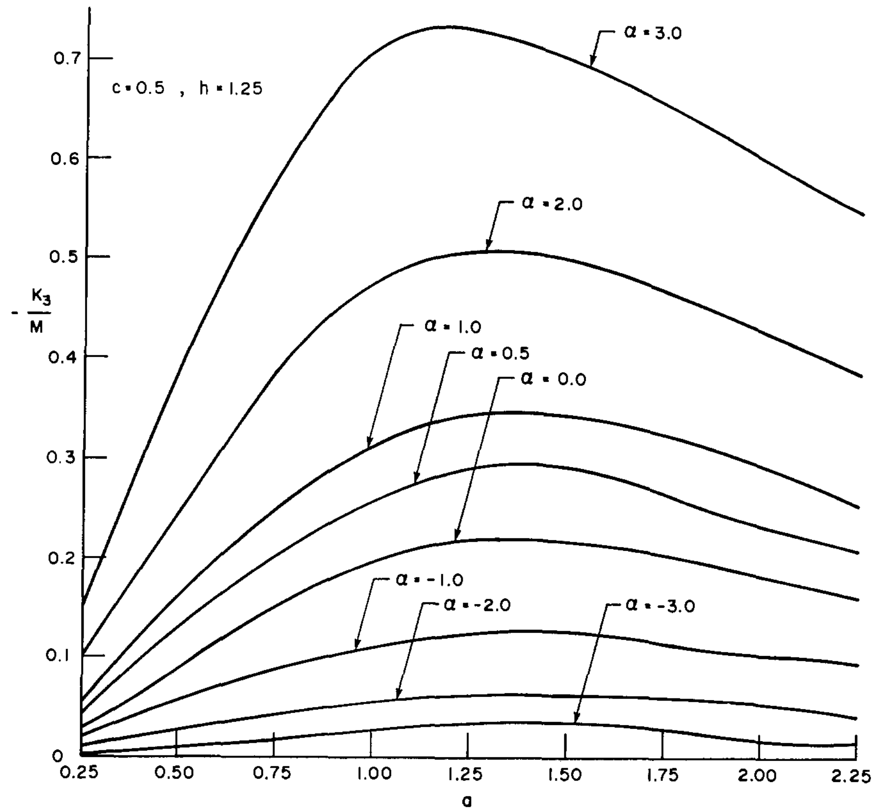

Fig. 5. Variation of $\frac{-K_3}{M}$ with a for $\alpha = 3.0, 2.0, 1.0, 0.5, 0.0, -1.0, -2.0, -3.0$ and $c = 0.5, h = 1.25$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{r^2 J_1(r\xi)\,\mathrm{d}r}{(t^2 - r^2)^{1/2}} = t \sin(\xi t) \tag{36}$$

to yield

$$\phi(t) + \int_0^a L_1(t,u)\psi(u)\,\mathrm{d}u + \int_0^b K_2(u,t)\phi(u)\,\mathrm{d}u = \frac{4t\sqrt{c}\,\omega_0}{\pi c^p}, \quad 0 < t < b \tag{37}$$

where

$$K_2(u,t) = \frac{2}{\pi} \int_0^\infty \left(\frac{K_p(c\xi)}{K_{p-1}(c\xi)} - 1 \right) \sin(u\xi) \sin(t\xi)\,\mathrm{d}\xi, \tag{38}$$

with $L_1(u,t)$ as defined by equation (31). Equation (36) is the derivative of equation (29) and is obtained using the results in Watson [14, section 3.56].

Equations (30) and (37) are two simultaneous Fredholm integral equations. Solving these numerically we can find numerical values of $\phi(t)$ and $\psi(t)$ as functions of t. The expression for the couple required to sustain the rotation for the disk can be computed from the formula

$$M = -2\pi \int_0^b r^2 \sigma_{\theta z}(r,0)\,\mathrm{d}r. \tag{39}$$

We find from equation (7) that

$$\sigma_{\theta z}(r,0) = \mu_1 c^{1-p} \frac{\partial}{\partial r} \int_0^\infty [A(\xi)K_{p-1}(c\xi) - B(\xi)I_{p-1}(c\xi)] J_0(r\xi)\,\mathrm{d}\xi, \quad 0 < r < b \tag{40}$$

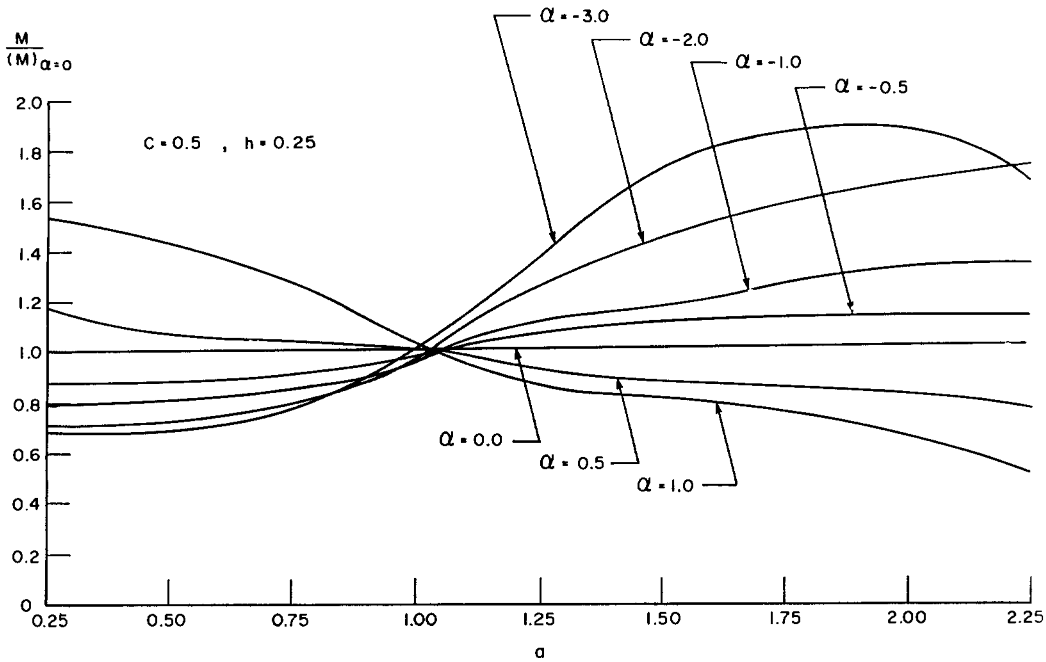

Fig. 6. Variation of $\frac{M}{M_{\alpha=0}}$ with a for $\alpha = 1.0, 0.5, 0.0, -0.5, -1.0, -3.0$ with $c = 0.5, h = 0.25$.

Making use of equation (23), we get from equation (40) that

$$\sigma_{\theta z}(r,0) = \frac{\mu_1 c^{1-p}}{\sqrt{c}} \frac{\partial}{\partial r} \int_r^b \frac{\phi(u)\,\mathrm{d}u}{\sqrt{u^2 - r^2}}, \quad 0 < r < b \tag{41}$$

Substituting equation (41) into equation (39) and integrating by parts, we get

$$M = \frac{4\pi\mu_1 c^{1-p}}{\sqrt{c}} \int_0^b u\phi(u)\,\mathrm{d}u. \tag{42}$$

With the help of equations (23), (24), (26a), and (26b), the expression (7) for the shear stress at $z = h$ can be written in the following form:

$$\begin{aligned} \sigma_{\theta z}(r,h) \;=\; & -\frac{\mu_1 (c+h)^{1-p}}{(c+h)^{1/2}} \left[\frac{a\psi(a)}{2r\sqrt{r^2 - a^2}} - \frac{1}{2r} \int_0^a \frac{[t\psi(t)]'\,\mathrm{d}t}{(r^2 - t^2)^{1/2}} \right. \\ & + \int_0^a \psi(t) \int_0^\infty (c+h)\xi^2 K_{p-1}[(c+h)\xi] \\ & \times \left\{ \frac{K_{p-1}[(c+h)\xi] I_{p-1}(c\xi)}{K_{p-1}(c\xi)} - I_{p-1}[(c+h)\xi] + \frac{1}{2\xi(c+h)J_{p-1}[(c+h)\xi]} \right\} \\ & \times J_1(r\xi) \left(\frac{\sin(\xi t)}{\xi t} - \cos(\xi t) \right) \mathrm{d}\xi\,\mathrm{d}t \end{aligned}$$

$$+\frac{\sqrt{(c+h)}}{\sqrt{c}}\int_0^b \phi(u)\int_0^\infty \frac{\xi K_{p-1}[(c+h)\xi]}{K_{p-1}(c\xi)}\sin(\xi u)J_1(r\xi)\,\mathrm{d}\xi\,\mathrm{d}u\Bigg], \quad a<r \tag{43}$$

The stress-intensity factor at $r = a, z = h$ is defined by

$$K_3 = \lim_{r\to a^+}\sqrt{2(r-a)}\sigma_{\theta z}(r,h). \tag{44}$$

Making use of equations (43) and (44), we find that

$$K_3 = \frac{-\mu_1(c+h)^{1-p}\psi(a)}{2(c+h)^{1/2}\sqrt{a}}. \tag{45}$$

CASE 1:

If we take $\alpha = 0$, then $p = \frac{1}{2}$ and we get the corresponding problem for the homogeneous medium. Setting $\alpha = 0$, equations (30), (37), (42), and (45) reduced to the following equations:

$$\psi(t) - 2\int_0^b L(u,t)\phi(u)\,\mathrm{d}u + \int_0^a K(u,t)\psi(u)\,\mathrm{d}u = 0, \quad 0<t<a, \tag{46}$$

$$\phi(t) + \int_0^a L(t,u)\psi(u)\,\mathrm{d}u = \frac{4\omega_0 t}{\pi}, \quad 0<t<b, \tag{47}$$

$$M = 4\pi\mu_1\int_0^b u\phi(u)\,\mathrm{d}u, \tag{48}$$

$$K_3 = -\frac{\mu_1\psi(a)}{2\sqrt{a}}, \tag{49}$$

where

$$L(t,u) = \frac{2}{\pi}\int_0^\infty e^{-\xi h}\sin(\xi t)\left(\frac{\sin(\xi u)}{\xi u} - \cos(\xi u)\right)\,\mathrm{d}\xi, \tag{50}$$

$$K(u,t) = -\sqrt{ut}\int_0^\infty \xi e^{-2\xi h}J_{3/2}(u\xi)J_{3/2}(t\xi)\,\mathrm{d}\xi. \tag{51}$$

In getting equations (46) and (47), we used the relationships that

$$K_{\pm 1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}, \tag{52}$$

and

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cosh(z). \tag{53}$$

This case is discussed in the book by Kassir and Sih [17, pp. 230-233]. If we compare our results with those found in that book, we find that the equations (7.41b) of Kassir and

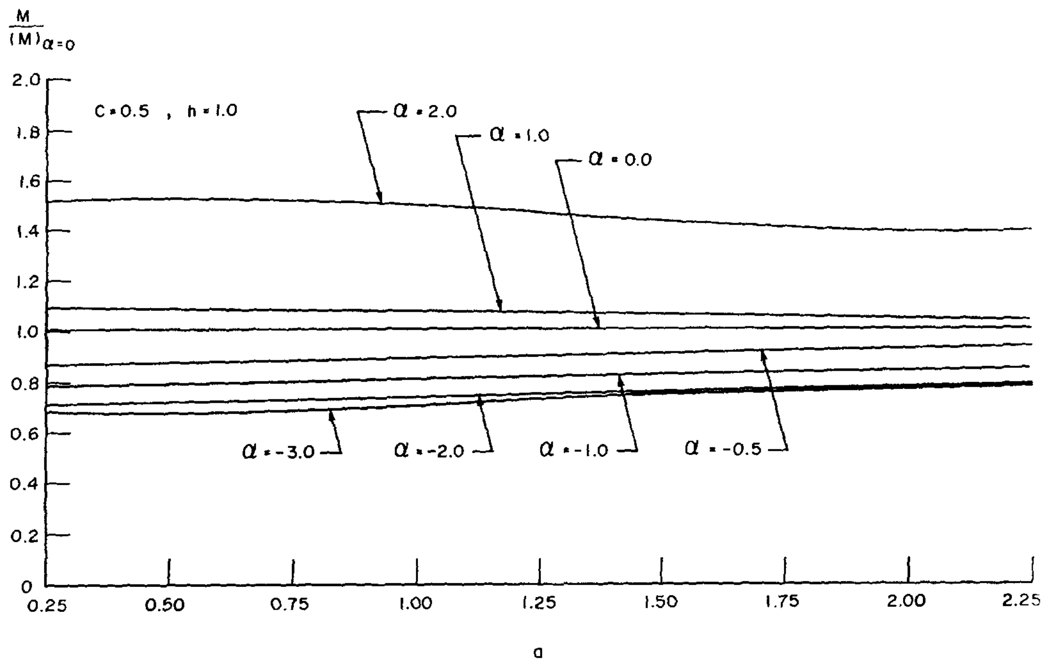

Fig. 7. Variation of $\frac{M}{M_{\alpha=0}}$ with a for $\alpha = 1.0, 0.5, 0.0, -0.5, -1.0, -3.0$ and $c = 0.5, h = 1.0$.

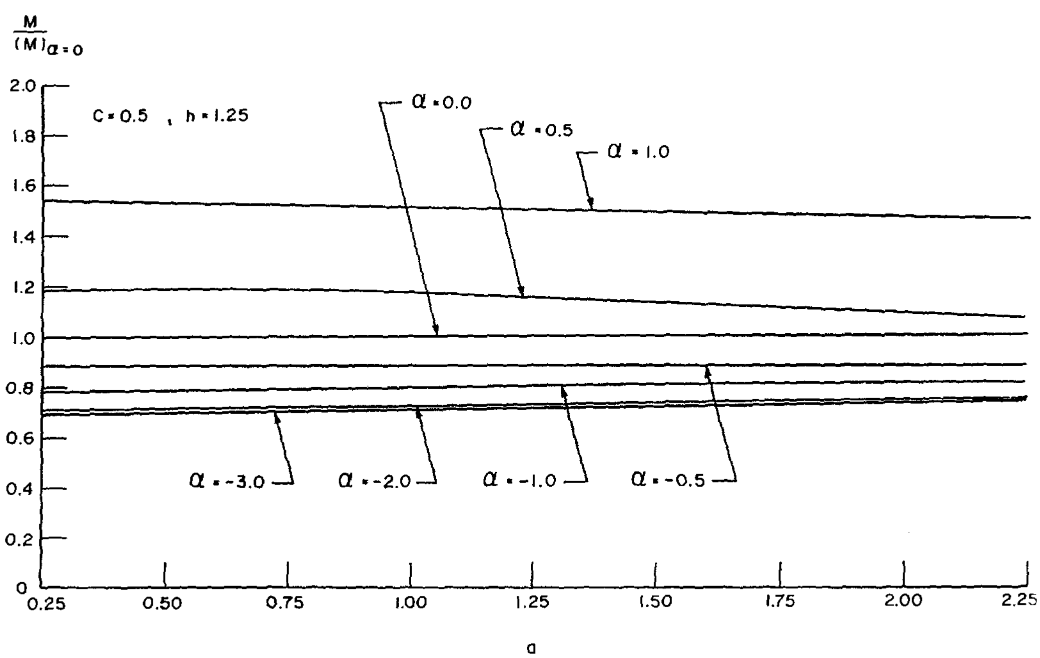

Fig. 8. Variation of $\frac{M}{M_{\alpha=0}}$ with a for $\alpha = 1.0, 0.5, 0.0, -0.5, -1.0, -3.0$ and $c = 0.5, h = 1.25$.

Sih [17, pp. 232] differ from ours. In place of $L(s,t)$ we read $L(t,s)$ and consequently the numerical results for this case as discussed in the book by Kassir and Sih [17] or Low [16] are correspondingly different.

CASE II:

Letting $a \to 0$ removes the crack from the semi-infinite medium. The solution of the problem then reduces to the torsion of a non-homogeneous solid by a rigid disk as discussed by Chuaprasert and Kassir [9]. Equation (37) reduces to the integral equation

$$\phi(t) + \int_0^b K_2(u,t)\phi(u)\,\mathrm{d}u = \frac{4t\omega_0\sqrt{c}}{\pi c^p}, \quad 0 < t < b, \tag{54}$$

the kernel of which is the same as discussed by Chuaprasert and Kassir [9, p. 706, equation 11].

CASE III: $a \to 0$ and $\alpha \to 0$

This is the classical Reissner-Sagoci static problem for a homogeneous isotropic elastic half-space. In this case, we find from (38) that

$$K_2(u,t) = 0 \tag{55}$$

and hence from equation (54) we have

$$\phi(t) = \frac{4t\omega_0}{\pi}, \quad 0 < t < b, \tag{56}$$

and with the help of equation (42) we get

$$M_\infty = \frac{16}{3}\,\omega_0 b^3 \mu, \tag{57}$$

which is in agreement with that obtained by Reissner and Sagoci [2].

5. Conclusions

In this paper, the final analysis was reduced to solving numerically the simultaneous Fredholm integral equations (30) and (37) for the function $\psi(t)$ and $\phi(t)$ and then for the torque M and the stress intensity factor K_3 from equations (42) and (45). Each of the integral equations (30) and (37) were discretized using a $16 \times 16, 20 \times 20$ and 24×24 grid at which point the results reached an acceptable 3-4 significant digit accuracy. The kernels (31), (32) and (38) have no singularities and expressions of the 0/0 type were, of course, replaced by the corresponding limits. The only modification required was to replace the unknown function $\psi(t)$ by $\chi(t) = t\psi(t)$, and then to reformulate the equations correspondingly and to work with $\chi(t)$ and $\phi(t)$ consistently throughout. Details of discretization mentioned above are as follows:

The $(0,b)$ and $(0,a)$ intervals were divided into $n(=16, 20, 24)$ subintervals and the value of $\chi(t)$ and $\phi(t)$ at the center of each of these subintervals become the new set of unknowns (the two equations were combined into a single one for numerical purposes). The integral over each subinterval *e.g.* $\int_{\mu_i-\Delta/2}^{\mu_i+\Delta/2} \phi(u)K_2(u,t_j)\,du$ was replaced by $\phi(\mu_i)\int_{\mu_i-\Delta/2}^{\mu_i+\Delta/2} K_2(u,t_j)\,du$

where μ_i and t_j are the center of subinterval values and the actual integration of K_2 was done analytically requiring integration of sin and $J_{3/2}$ functions only. Each of the integrals (31), (32), (38) was determined numerically with respect to ξ (subsequent to the u integration). Since the Bessel functions of all these integrals decreases either exponentially or at least with respect to a second power of ξ (which arises due to the previous u-integration) further amplified by the oscillating behaviour of the remaining part of the function, it was possible to simply replace ∞ by a large number (verifying numerically that further increase of the number would not change the results).

A plot of $\pi\psi(a)/4a\omega_0$ versus a for $h = 0.25, 0.50, 1, 1.25$, and $\alpha = 0, b = 1$, is exhibited in Fig. 2. Note that curves peak at $a \cong 1$. A comparison of these results with Low [16] and Kassir and Sih [17, p. 233] show that the results differ but are of the same trend. The reason for this has already been discussed in Case 1. To illustrate the behavior of the curves, Figs 3–5 show the results for $-K_3/M$ for $b = 1, c = 0.5$ for various values of the parameters α, a, h. In Figs 6–8, the results for $M/(M)_{\alpha=0}$ are also shown for $b = 1, c = 0.5$ for various values of h.

It is clear from the figures that the trends for $h < 1$ and $h \geqslant 1$ are different and inclusion of these figures is essential for an understanding of the problem.

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